III. *On the pulsations of spheres in an elastic medium.* By A. H. Leahy, M.A.

1. The motion due to the pulsations of spheres of the same period of pulsation in an incompressible fluid has been investigated by Professor Bjerknes of Christiania*, by whom the following results have been obtained. If the pulsations of two spheres are in the same phase of vibration, there will be an apparent force on each of the bodies, which varies according to the law of the inverse square of the distance, and tends to make the spheres approach one another; but, if the pulsations are in phases differing by half a complete period, there will be a force tending to repel the spheres from one another, and varying according to the same law These results have been experimentally verified, and similar effects have been shewn by some experiments, described in the *Journal of Telegraph Engineers* for 1882, to hold in air. An apparatus shewing these attractive and repulsive effects, together with several other "inverse analogies," to use Dr Bjerknes' phrase, between electro magnetic effects and pulsations under water of spheres and cylinders was exhibited at the Paris exhibition†.

2. These phenomena, together with several others of a kindred character, may be explained by the following general considerations. Suppose a periodic force of the nature of surface tensions or pressures to be acting on a sphere, whose centre is fixed in space, and which is itself pulsating with a simple harmonic motion. Then, since the magnitude of the force which acts upon the body varies as the superficial area, it is clear that the effect of the force will be greatest, when the surface of the body is greatest. If therefore the force is a simple harmonic function of the time, and has the same period as that of the pulsations of the body, it is clear that its effect during one complete oscillation will be to urge the body in the direction in which the force acted when the area of the sphere was a maximum. For, considering any two instants the time between which is half a complete period, it is clear that the force at each of these instants will be the same in magnitude and opposite in direction; so that the resultant effect will be to urge the body in the direction which the force had when the superficial area of the sphere was the greater. Thus we have only to consider the effect of the force during that half period when the sphere is greater than its mean value; i.e. than its value at a time midway between the instants of greatest contraction and expansion. Let τ be the time when the sphere is greatest. Then, if $2p$ is the complete period, we shall only have to

* See the Reports of the *Proceedings of the Scientific Society of Christiania*, 1875, and the *Repertorium der Mathematik von Königsberger und Zeuner*, 1876, p. 268.

† See *La Lumière Electrique*, 5th Oct. and 9th Nov. 1881, and *Engineering*, 1882.

consider the force between the instants $\tau + \frac{p}{2}$ and $\tau - \frac{p}{2}$. But if τ' is the time when the force on the sphere is zero, where τ' lies between $\tau + \frac{p}{2}$ and $\tau - \frac{p}{2}$, it is clear that the force at the instants $\tau' + \alpha$ and $\tau' - \alpha$ are the same in magnitude and opposite in direction. Also, if $\tau' - \tau$ is positive, the effect of the force at the time $\tau' + \alpha$ will be less than at the time $\tau' - \alpha$; since the area of the body which is acted on is less at the former instant than at the latter. Thus the action on the sphere during the period between the instants $\tau - \frac{p}{2}$ and τ' will exceed the action in the opposite direction during the period between τ' and $\tau + \frac{p}{2}$, and the resultant action during a complete oscillation will be the same in direction as at the time τ, when the area of the sphere was a maximum. A similar result will follow if $\tau' - \tau$ is negative.

We have therefore, in order to determine the direction in which a periodic force of the character described urges a pulsating sphere, merely to determine the direction which the force has when the area of the sphere is a maximum. Now, in the case of two spheres A and B pulsating with their centres fixed in an incompressible fluid; it can easily be seen that the change of pressure due to the pulsations of A increases with the time differential of the velocity along the radius vector from the centre of A. The action on B due to this change of pressure is of course greater on that side which faces A than on the opposite side, and the force will therefore be a repulsion when the velocity due to the pulsations of A is increasing, and an attraction when the velocity is diminishing. Now when the volume of A is greater than its mean value the velocity is diminishing; hence, if the pulsations of A and B are in the same phase of vibration, there will be an attractive force on B when its volume is greatest, and the general effect of the changes of pressure due to A's pulsation will be an attraction towards A. Similarly, if the pulsations are in opposite phases, the effect will be a repulsion.

But if these changes of volume are executed in a medium having properties similar to those of the ether, in which the vibrations producing the sensation of light are supposed to be propagated, the results which have been given above will not continue to hold. For, in this case there will be no flux at the surface of B, if the displacements are not large; and the force will not depend upon the velocity, or upon the changes of velocity, but upon the absolute deformations. If the waves of displacement are long, compared with the distance between A and B, the medium will be compressed as A expands; and the effect at the surface of B will be a repulsion if the volume of A is greater than its mean value. Thus the effects produced in an incompressible fluid will be reversed if the oscillations are performed in an elastic medium, and like phases of pulsation will give rise to a repulsion and unlike phases to an attraction on the pulsating bodies. This way of looking at the problem appears to indicate, that, if spheres are pulsating in an elastic medium, the period of pulsation being such as to give rise to waves which are long, compared with the distances between the spheres, the results obtained by Professor Bjerknes will be reversed. If the distance of B from A exceed a quarter wave length it is evident that this result will not be true. Supposing for example, that the medium under consideration is the same as that in which light-vibrations are performed, we shall have, taking the approximate velocity of propagation to be 200,000 miles a second, a wave length of 200 miles corresponding to 1000 vibrations in a second. Thus for all distances of A and B at which any sensible effect can be observed we can take the phase of the vibration to be the same; but if the distance exceeds 50 miles our argument does not apply.

3. These considerations are founded on a principle which seems to underlie many cases of differential action; namely, that if a body be acted on by a force F, where F is a simple harmonic function of the time; and, if the action of the force on the body due to variations in the position, magnitude, or shape of the body be expressed by FF', where F' is also a simple harmonic function of the time of the same period as F; then the effect of the force on the body will be to urge it in that direction which F had when F' was a maximum. This principle can also be extended to the case where F' is any periodic function of the same period as F, provided that F' has only one maximum value during the complete period $2p$, and also satisfies the condition $F'(\tau+\alpha)=F'(\tau-\alpha)$, where τ is the time when F' is a maximum. The truth of this principle can be established by the same considerations as those employed in § 2; since we shall have $F'(t)$ diminishing, as the numerical value of $t-\tau$ increases from zero to p; the complete period being $2p$; and the whole of the argument at the beginning of § 2 will apply. As an example of differential action which can be treated by the principle just enunciated may be mentioned that of a body placed in a field of force, where the force at any point has for components $L\sin\frac{\pi t}{p}$, $M\sin\frac{\pi t}{p}$, $N\sin\frac{\pi t}{p}$, where L, M, N are functions of the co-ordinates of the point. Let the body be constrained by some independent cause to move, so that at the time t its position is such that the force acting on it has for components $L\sin\frac{\pi t}{p}$, $M\sin\frac{\pi t}{p}$, $N\sin\frac{\pi t}{p}$, where L, M, N are periodic functions of the time, of period equal to $2p$, which have only one maximum value during that period, and satisfy the functional equation $f(\tau+\alpha)=f(\tau-\alpha)$, where τ is the time when f is a maximum. It will then be found that the action of any component $L\sin\frac{\pi t}{p}$ will be to urge the body in that direction in which the component acted when L was a maximum.

4. These considerations do not however give the law of the action of the force, either in this case, or in the case of a pulsating body which was mentioned in § 2. In order to completely investigate the mutual action of two pulsating bodies in an elastic medium it will be necessary to find the displacement at any point due to their joint effect, and it will be found that the law of attraction in the case of unlike phases, and of repulsion in the case of like phases will be that of the inverse square of the distance to the first order of approximation. The term of next order of importance will always be a repulsion and will vary according to the law of the inverse cube. In the following work I propose to establish these results.

General expressions for a periodic and steady displacement symmetrical about an axis.

The equations of motion of an elastic medium* are

$$\left.\begin{aligned} r^2 \sin\theta \frac{d^2u}{dt^2} &= \frac{\lambda + 2\mu}{\rho} r^2 \sin\theta \frac{d\epsilon}{dr} + \frac{\mu}{\rho}\left(\frac{d\gamma}{d\theta} - \frac{d\beta}{d\phi}\right) \\ r \sin\theta \frac{d^2v}{dt^2} &= \frac{\lambda + 2\mu}{\rho} \sin\theta \frac{d\epsilon}{d\theta} + \frac{\mu}{\rho}\left(\frac{d\alpha}{d\phi} - \frac{d\gamma}{dr}\right) \\ r\frac{d^2w}{dt^2} &= \frac{\lambda + 2\mu}{\rho} \frac{1}{\sin\theta} \frac{d\epsilon}{d\phi} + \frac{\mu}{\rho}\left(\frac{d\beta}{dr} - \frac{d\alpha}{d\theta}\right) \end{aligned}\right\} \quad \ldots\ldots(1),$$

where u is the displacement along radius vector, v is the displacement along the tangent to the meridian tending from the pole, and w is the displacement along a parallel of longitude tending from the fixed meridian; r, θ, ϕ being the co-ordinates of a particle in its undisturbed position; λ and μ being the coefficients of elasticity of the medium, ρ its density, and ϵ, α, β, γ being defined by the equations

$$\left.\begin{aligned} \epsilon &= \frac{1}{r^2}\frac{d}{dr}(r^2u) + \frac{1}{r\sin\theta}\frac{d}{d\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{dw}{d\phi} \\ \alpha &= \frac{1}{r^2\sin\theta}\left\{\frac{d}{d\phi}(rv) - \frac{d}{d\theta}(rw\sin\theta)\right\} \\ \beta &= \frac{1}{\sin\theta}\left\{\frac{d}{dr}(rw\sin\theta) - \frac{du}{d\phi}\right\} \\ \gamma &= \sin\theta\left\{\frac{du}{d\theta} - \frac{d}{dr}(rv)\right\} \end{aligned}\right\} \quad \ldots\ldots(2).$$

The particular values of u, v, w depending upon $\frac{\lambda + 2\mu}{\rho}$, which are propagated with the velocity $\sqrt{\frac{\lambda + 2\mu}{\rho}}$, must make the terms vanish which depend upon $\frac{\mu}{\rho}$.

Thus we must have

$$\frac{d}{d\phi}(rv) - \frac{d}{d\theta}(rw\sin\theta) = r^2\sin\theta\frac{d\psi}{dr},$$

$$\frac{d}{dr}(rw\sin\theta) - \frac{du}{d\phi} = \sin\theta\frac{d\psi}{d\theta},$$

$$\frac{du}{d\theta} - \frac{d}{dr}(rv) = \frac{1}{\sin\theta}\frac{d\psi}{d\phi}.$$

Substituting in equations (1) we get $\frac{d^2}{dt^2}\left(\frac{d\psi}{dr}\right) = 0$, etc. Thus since ψ is essentially periodic, we must have $\frac{d\psi}{dr} = 0$, $\frac{d\psi}{d\theta} = 0$, $\frac{d\psi}{d\phi} = 0$, or $\alpha = \beta = \gamma = 0$.

* Lamé's *Elasticity*, Art. 84.

These conditions give

$$u = \frac{dF}{dr}, \quad rv = \frac{dF}{d\theta}, \quad rw \sin\theta = \frac{dF}{d\phi} \quad \text{..........(3)},$$

where F is a function of the same period as u, v, w. Hence $\epsilon = \nabla^2 F$, where

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2}{d\phi^2},$$

and equations (1) reduce to

$$\frac{d^2F}{dt^2} = \frac{\lambda + 2\mu}{\rho}\nabla^2 F \quad \text{..........(4)}.$$

Next taking the values of u, v, w which depend upon $\frac{\mu}{\rho}$, and which travel with velocity $\sqrt{\frac{\mu}{\rho}}$, we have $\epsilon = 0$, or

$$\frac{d}{dr}(r^2 u \sin\theta) + \frac{d}{d\theta}(rv\sin\theta) + \frac{d}{d\phi}(rw) = 0.$$

Hence we must have

$$\left.\begin{aligned} r^2\sin\theta \,.\, u &= \frac{dM}{d\phi} - \frac{dN}{d\theta} \\ r\sin\theta \,.\, v &= \frac{dN}{dr} - \frac{dL}{d\phi} \\ r \,.\, w &= \frac{dL}{d\theta} - \frac{dM}{dr} \end{aligned}\right\} \quad \text{..........(5)}.$$

Substituting in equations (1), we get

$$\frac{d}{d\theta}\left[\frac{d^2L}{dt^2} - \frac{\mu}{\rho}\left\{\nabla_1^2 L - \frac{2\sin\theta}{r^3}\frac{d}{d\theta}\left(\frac{M}{\sin\theta}\right) - \frac{2\cot\theta}{r^2}\frac{dM}{dr} - \frac{2}{r^3\sin^2\theta}\frac{dN}{d\phi}\right\}\right]$$
$$= \frac{d}{dr}\left\{\frac{d^2M}{dt^2} - \frac{M}{\rho}\left(\nabla_2^2 M - \frac{2\cot\theta}{r^2\sin^2\theta}\frac{dN}{d\phi}\right)\right\},$$

$$\frac{d}{d\phi}\left[\frac{d^2L}{dt^2} - \frac{\mu}{\rho}\left\{\nabla_1^2 L - \frac{2\sin\theta}{r^3}\frac{d}{d\theta}\left(\frac{M}{\sin\theta}\right) - \frac{2\cot\theta}{r^2}\frac{dM}{dr} - \frac{2}{r^3\sin^2\theta}\frac{dN}{d\phi}\right\}\right] = \frac{d}{dr}\left\{\frac{d^2N}{dt^2} - \frac{\mu}{\rho}\nabla_2^2 N\right\},$$

$$\frac{d}{d\phi}\left\{\frac{d^2M}{dt^2} - \frac{\mu}{\rho}\left(\nabla_2^2 M - \frac{2\cot\theta}{r^2\sin^2\theta}\frac{dN}{d\phi}\right)\right\} = \frac{d}{d\theta}\left\{\frac{d^2N}{dt^2} - \frac{\mu}{\rho}\nabla_2^2 N\right\};$$

where

$$\left.\begin{aligned} \nabla_1^2 &= \frac{d^2}{dr^2} + \frac{1}{r^2}\frac{d^2}{d\theta^2} + \frac{\cot\theta}{r^2}\frac{d}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2}{d\phi^2} \\ \nabla_2^2 &= \frac{d^2}{dr^2} + \frac{1}{r^2}\frac{d^2}{d\theta^2} - \frac{\cot\theta}{r^2}\frac{d}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2}{d\phi^2} \end{aligned}\right\} \quad \text{..........(6)}.$$

These results give

$$\left.\begin{aligned} \frac{d^2L}{dt^2} &= \frac{\mu}{\rho}\left\{\nabla_1^2 L - \frac{2\sin\theta}{r^3}\frac{d}{d\theta}\left(\frac{M}{\sin\theta}\right) - \frac{2\cot\theta}{r^2}\frac{dM}{dr} - \frac{2}{r^3\sin^2\theta}\frac{dN}{d\phi}\right\} \\ \frac{d^2M}{dt^2} &= \frac{\mu}{\rho}\left\{\nabla_2^2 M - \frac{2\cot\theta}{r^2\sin^2\theta}\frac{dN}{d\phi}\right\} \\ \frac{d^2N}{dt^2} &= \frac{\mu}{\rho}\nabla_2^2 N \end{aligned}\right\} \quad \text{..........(7)}.$$

5. In order to get a displacement symmetrical about an axis, we put $w=0$ and $\frac{d}{d\phi}=0$.

This will give

$$\left.\begin{aligned} u &= \frac{dF}{dr}-\frac{1}{r^2\sin\theta}\frac{dN}{d\theta} \\ v &= \frac{1}{r}\frac{dF}{d\theta}+\frac{1}{r\sin\theta}\frac{dN}{dr} \end{aligned}\right\} \quad\ldots\ldots(1),$$

where F satisfies the equation

$$\frac{d^2F}{dt^2}=\frac{\lambda+2\mu}{\rho}\left\{\frac{d^2F}{dr^2}+\frac{2}{r}\frac{dF}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right)\right\} \quad\ldots\ldots(2),$$

and N satisfies the equation

$$\frac{d^2N}{dt^2}=\frac{\mu}{\rho}\left\{\frac{d^2N}{dr^2}+\frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dN}{d\theta}\right)\right\} \quad\ldots\ldots(3).$$

Now, whatever be the forms of F and N, they can be expanded in series of Legendre's coefficients $A_nP_n(\mu)$, where $\mu=\cos\theta$ and A_n is a function of r and t only. Also, since $\sin\theta\frac{dP_n}{d\theta}$ can be expressed in terms of Legendre's coefficients, F and N can also be expressed in series of form $B_n\sin\theta\frac{dP_n}{d\theta}$, where B_n is a function of r and t only.

Put $F=\Sigma A_nP_n(\mu)$ and $N=\Sigma B_n\sin\theta\frac{dP_n(\mu)}{d\theta}$ and the equations (2) and (3) reduce to

$$\frac{d^2A_n}{dt^2}=\frac{\lambda+2\mu}{\rho}\left\{\frac{d^2A_n}{dr^2}+\frac{2}{r}\frac{dA_n}{dr}-\frac{n(n+1)}{r^2}A_n\right\},$$

$$\frac{d^2B_n}{dt^2}=\frac{\mu}{\rho}\left\{\frac{d^2B_n}{dr^2}-\frac{n(n+1)}{r^2}B_n\right\}.$$

Let us now suppose the displacement to be oscillatory, and steady, of period equal to $\frac{2\pi}{p}$. We shall get, on this supposition, if $k^2=\frac{\rho p^2}{\lambda+2\mu}$, $h^2=\frac{\rho p^2}{\mu}$,

$$A_n=f_n(ikr)\,e^{ipt},$$

$$B_n=F_n(ihr)\,e^{ipt},$$

where f_n, F_n are the solutions of the equations

$$\frac{d^2f_n}{d(kr)^2}+\frac{2}{kr}\frac{df_n}{d(kr)}+\left\{1-\frac{n(n+1)}{(kr)^2}\right\}f_n=0 \text{ and } \frac{d^2F_n}{d(hr)^2}+\left\{1-\frac{n(n+1)}{(hr)^2}\right\}F_n=0.$$

The solutions of these equations are

$$\left.\begin{aligned} f_n(ikr) &= A_nr^n\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}e^{ikr}+A_n'r^n\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}e^{-ikr} \\ F_n(ihr) &= B_nr^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}e^{ihr}+B_n'r^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}e^{-ihr} \end{aligned}\right\} \quad\ldots\ldots(4).$$

Hence, if f_n and F_n have the meanings given above;

$$\left.\begin{aligned} u &= \Sigma P_n(\mu)\left\{\frac{d.f_n(ikr)}{dr}+\frac{n(n+1)}{r^2}F_n(ihr)\right\}e^{ipt} \\ v &= \Sigma\frac{1}{r}\frac{dP_n(\mu)}{d\theta}\left\{f_n(ikr)+\frac{d.F_n(ihr)}{dr}\right\}e^{ipt}\end{aligned}\right\}\text{..........(5)},$$

with two similar terms obtained by changing the sign of i, will give the complete value of a periodic and steady vibration, which is symmetrical about an axis.

6. The series $f_n(ikr)$ and $F_n(ihr)$ can be expanded in powers of kr and hr respectively.

For putting $r^2=z$, we have $\qquad \dfrac{1}{r}\dfrac{d}{dr}=2\dfrac{d}{dz},$

therefore

$$\left.\begin{aligned} f_n(ikr) &= A_n z^{\frac{n}{2}}.\,2^{n+1}\left(\frac{d}{dz}\right)^{n+1}e^{ik\sqrt{z}} \\ F_n(ihr) &= B_n z^{\frac{n+1}{2}}.\,2^{n+1}\left(\frac{d}{dz}\right)^{n+1}e^{ih\sqrt{z}}\end{aligned}\right\}\text{..........(1)},$$

with other terms obtained by changing the sign of i.

Hence, if when $n-s$ is negative, $\dfrac{2n-2s!}{n-s!}$ means $(-1)^{n-s}.\dfrac{s-n!}{2s-2n!}$,

$$\left.\begin{aligned} f_n(ikr) &= (-1)^n A_n ikr^{-(n+1)}\left\{\frac{2n!}{2^n.n!}\ldots\ldots+(-1)^s\frac{2n-2s!}{2^n.s!.n-s!}(ikr)^{2s}+\frac{2^{n+1}.s+1!}{2s+2!\,s-n!}(ikr)^{2s+1}+\ldots\right\} \\ F_n(ihr) &= (-1)^n B_n.\,ihr^{-n}\left\{\frac{2n!}{2^n.n!}\ldots\ldots+(-1)^s.\frac{2n-2s!}{2^n.s!\,n-s!}(ihr)^{2s}+\frac{2^{n+1}.s+1!}{2s+2!\,s-n!}(ihr)^{2s+1}\ldots\ldots\right\}\end{aligned}\right\}(2),$$

the highest odd powers of ikr and of ihr in the series being $(ikr)^{2n+1}$ and $(ihr)^{2n+1}$.

Hence we get

$$\left.\begin{aligned} u_n =& (-1)^{n+1}A_nP_n(\mu).r^{-(n+2)}.ik\left\{\frac{2n!\,(n+1)}{2^n.n!}\ldots+(-1)^{s+1}.\frac{2n-2s!\,(2s-n-1)}{2^n.s!\,n-s!}(ikr)^{2s}-\frac{2^{n+1}.s+1!\,(2s-n)}{2s+2!\,s-n!}(ikr)^{2s+1}\ldots\right\} \\ &+(-1)^nB_nP_n(\mu)r^{-(n+2)}.ih\left\{\frac{2n!\,n(n+1)}{2^n.n!}\ldots+(-1)^s.\frac{2n-2s!\,n(n+1)}{2^n.s!\,n-s!}(ihr)^{2s}+\frac{2^{n+1}.s+1!\,n(n+1)}{2s+2!\,s-n!}(ihr)^{2s+1}\ldots\right\} \\ v_n =& (-1)^nA_n\frac{dP_n(\mu)}{d\theta}.r^{-(n+2)}.ik\left\{\frac{2n!}{2^n.n!}\ldots+(-1)^s.\frac{2n-2s!}{2^n.s!\,n-s!}(ikr)^{2s}+\frac{2^{n+1}.s+1!}{2s+2!\,s-n!}(ikr)^{2s+1}\ldots\right\} \\ &+(-1)^{n+1}.B_n\frac{dP_n(\mu)}{d\theta}r^{-(n+2)}.ih\left\{\frac{2n!\,n}{2^n.n!}\ldots+(-1)^{s+1}.\frac{2n-2s!\,(2s-n)}{2^n.s!\,n-s!}(ihr)^{2s}-\frac{2^{n+1}.s+1!\,(2s-n-1)}{2s+2!\,s-n!}(ihr)^{2s+1}\ldots\right\}\end{aligned}\right\}(3).$$

7. Let us now suppose that the medium extends without limit. When r is very great $r^{n+1}\left(\dfrac{1}{r}\dfrac{d}{dr}\right)e^{ikr}=(ik)^{n+1}e^{ikr}$. Hence at a very great distance from the origin the coefficients of $P_n(\mu)$ and $\dfrac{dP_n(\mu)}{d\theta}$ in equation (5), § 5 become

$$u_n = \frac{ik}{r}\{C_n e^{i(pt+kr)} - C_n' e^{i(pt-kr)}\} + \frac{n(n+1)}{r^2}\{D_n e^{i(pt+hr)} + D_n' e^{i(pt-hr)}\},$$

$$v_n = \frac{1}{r^2}\{C_n e^{i(pt+kr)} + C_n' e^{i(pt-kr)}\} + \frac{ih}{r}\{D_n e^{i(pt+hr)} - D_n' e^{i(pt-hr)}\}.$$

The terms $e^{i(pt+hr)}$ and $e^{i(pt+kr)}$ represent disturbances travelling inwards. Hence, if the disturbance is to be zero at infinity, we have $C_n = 0$, $D_n = 0$; and the amount of the displacement at any point in a medium extending to infinity and bounded internally by a surface vibrating in any assigned manner is given by two functions u and v; of the form given in equation (5), § 5, where

$$f_n(ikr) = A_n r^n \left(\frac{1}{r}\frac{d}{dr}\right) e^{-ikr},$$

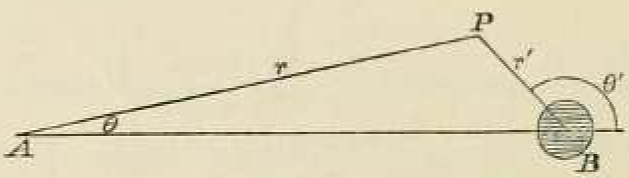
$$F_n(ihr) = B_n r^{n+1} \left(\frac{1}{r}\frac{d}{dr}\right) e^{-ihr}.$$

To investigate the disturbance produced by the presence of a small fixed sphere on the axis of symmetry, if there is no slipping at the surface of the sphere.

8. We shall now investigate the disturbance produced by the presence of a small fixed sphere on the axis of symmetry, where the disturbance if the sphere were removed would be given by the expressions

$$u = \text{displacement along } AP = \Sigma P_n(\mu)\left\{\frac{df_n}{dr} + \frac{n(n+1)}{r^2}F_n\right\}e^{ipt},$$

$$v = \text{displacement perpendicular to } AP = \Sigma \frac{1}{r}\frac{dP_n(\mu)}{d\theta}\left\{f_n + \frac{dF_n}{dr}\right\}e^{ipt}.$$



where
$$f_n = A_n r^n \left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{-ikr},$$

$$F_n = A_n' r^{n+1} \left(\frac{1}{r}\frac{d}{dr}\right)^{n+1} e^{-ihr},$$

the waves being supposed to be long compared with the distance between the origin A and the centre of sphere B. Let radius of B be b, and let distance between A and centre of B be c. In order to express the conditions that the displacement should be zero at the surface of B, we must find the resolved parts along and perpendicular to BP of the displacement given above by u and v.

Taking the coefficient of $P_n(\mu)$,

$$\left.\begin{aligned} u_n' &= \text{displacement along } BP = u_n \cdot \frac{r' + c\cos\theta'}{r} + v_n \frac{c\sin\theta'}{r} \\ v_n' &= \text{displacement perpendicular to } BP = -u_n \frac{c\sin\theta'}{r} + v_n \frac{r' + c\cos\theta'}{r} \end{aligned}\right\} \ldots\ldots\ldots\ldots (1).$$

Taking the leading terms only of expressions for u_n, v_n given in equations (3) of paragraph 6; and omitting the time factor e^{ipt}, we get

$$\left.\begin{aligned} u_n' &= (-1)^n . ic . \frac{2n!}{2^n . n!} \cdot \frac{A_n k - A_n' h}{r^{n+3}} \left\{ \sin\theta' \frac{dP_n(\mu)}{d\theta} - (n+1)\left(\cos\theta' + \frac{r'}{c}\right) P_n(\mu) \right\} \\ v_n' &= (-1)^n . ic . \frac{2n!}{2^n . n!} \cdot \frac{A_n k - A_n' h}{r^{n+3}} \left\{ \left(\cos\theta' + \frac{r'}{c}\right) \frac{dP_n(\mu)}{d\theta} + (n+1) P_n(\mu) . \sin\theta' \right\} \end{aligned}\right\} \ldots\ldots\ldots (2).$$

We shall have to express u_n' and v_n' in series involving Legendre's functions of μ', where $\mu' = \cos\theta'$ and their differential coefficients with respect to θ'.

This operation can be facilitated by the following transformations.

By a known formula*,

$$P_n(\mu) = (-1)^n . \frac{r^{n+1}}{n!} \frac{d^n}{dc^n} \cdot \frac{1}{r},$$

therefore, differentiating $\frac{1}{r}$ n times with respect to c,

$$\frac{P_n(\mu)}{r^{n+1}} = \frac{1}{c^{n+1}} \left\{ 1 - (n+1) P_1(\mu') \frac{r'}{c} + \frac{(n+1)(n+2)}{2!} P_2(\mu') . \frac{r'^2}{c^2} \ldots \right\} \ldots\ldots\ldots\ldots\ldots (3).$$

Differentiating this result (first) with respect to θ' keeping r' constant, and then with respect to r', keeping θ' constant, we get

$$\left.\begin{aligned} &\frac{1}{r^{n+3}} \left\{ \sin\theta' \frac{dP_n(\mu)}{d\theta} - \left(\cos\theta' + \frac{r'}{c}\right)(n+1) P_n(\mu) \right\} \\ &\quad = -\frac{1}{c^{n+3}} \left\{ (n+1) P_1(\mu') - \frac{(n+1)(n+2)}{1!} P_2(\mu') \frac{r'}{c} + \frac{(n+1)(n+2)(n+3)}{2!} P_3(\mu') \frac{r'^2}{c^2} \ldots \right\} \\ &\frac{1}{r^{n+3}} \left\{ \left(\cos\theta' + \frac{r'}{c}\right) \frac{dP_n(\mu)}{d\theta} + (n+1)\sin\theta' P_n(\mu) \right\} \\ &= -\frac{1}{c^{n+3}} \left\{ (n+1) \frac{dP_1(\mu')}{d\theta'} - \frac{(n+1)(n+2)}{2!} \frac{dP_2(\mu')}{d\theta'} \frac{r'}{c} + \frac{(n+1)(n+2)(n+3)}{3!} \frac{dP_3(\mu')}{d\theta'} \frac{r'^2}{c^2} \ldots \right\} \end{aligned}\right\} \ldots (4).$$

Reducing the values of u_n', v_n' given in result (2) above by means of relations (4) we get

$$u_n' = (-1)^n . ic . \frac{2n!}{2^n . n!} \left\{ (n+1) P_1(\mu') - \frac{(n+1)(n+2)}{1!} P_2(\mu') \frac{r'}{c} + \ldots \right\} \frac{A_n k - nA_n' h}{c^{n+3}},$$

$$v_n' = (-1)^n . ic . \frac{2n!}{2^n . n!} \left\{ (n+1) \frac{dP_1(\mu')}{d\theta'} - \frac{(n+1)(n+2)}{2!} \frac{dP_2(\mu')}{d\theta'} \frac{r'}{c} + \ldots \right\} \frac{A_n k - nA_n' h}{c^{n+3}};$$

* See Maxwell's *Electricity*, Vol. I. art. 132, equation (28).

or

$$\left.\begin{aligned} u_n' &= (-1)^{n+1}.\,ic.\frac{2n!}{2^n.(n!)^2}\sum_{m=0}^{m=\infty}(-1)^m.\frac{m+n!}{m-1!}P_m(\mu').\frac{r'^{m-1}}{c^{m-1}}.\frac{A_nk-nA_n'h}{c^{n+3}} \\ v_n' &= (-1)^{n+1}.\,ic.\frac{2n!}{2^n.(n!)^2}\sum_{m=0}^{m=\infty}(-1)^m.\frac{m+n!}{m!}\frac{dP_m(\mu')}{d\theta'}.\frac{r'^{m-1}}{c^{m-1}}.\frac{A_nk-nA_n'h}{c^{n+3}} \end{aligned}\right\}\ldots\ldots\ldots\ldots(5).$$

In order that the surface conditions may be satisfied, we shall have to take $u_n'+(u_n')$ for the whole displacement along BP; and $v_n'+(v_n')$ for the whole displacement perpendicular to BP; where (u_n') and (v_n') are to be of such magnitude as to cause the whole displacement due to u_n, v_n to be zero at the surface of the sphere B. These terms (u_n') (v_n') must be of the form given in equations (3), § 6, namely,

$$\left.\begin{aligned} (u_n') &= \sum_{m=0}^{m=\infty}(-1)^{m+1}P_m(\mu')B_m.(ik)r'^{-(m+2)}\left\{\frac{2m!(m+1)}{2^m.m!}-\frac{2m-2!(m-1)}{2^m.1!\,m-1!}(ikr')^2+\frac{2m-4!(m-3)}{2^m.2!\,m-2!}(ikr')^4\ldots\right\} \\ &+\sum_{m=0}^{m=\infty}(-1)^m P_m(\mu').B_m'(ih)r'^{-(m+2)}\left\{\frac{2m!\,m(m+1)}{2^m.m!}-\frac{2m-2!\,m(m+1)}{2^m.1!\,m-1!}(ihr')^2+\frac{2m-4!\,m(m+1)}{2^m.2!\,m-2!}(ihr')^4\ldots\right\} \\ (v_n') &= \sum_{m=1}^{m=\infty}(-1)^m.\frac{dP_m(\mu')}{d\theta'}.B_m.(ik)r'^{-(m+2)}\left\{\frac{2m!}{2^m.m!}-\frac{2m-2!}{2^m.1!\,m-1!}(ikr')^2+\frac{2m-4!}{2^m.2!\,m-2!}(ikr')^4\ldots\right\} \\ &+\sum_{m=1}^{m=\infty}(-1)^{m+1}\frac{dP_m(\mu')}{d\theta'}.B_m'.(ih)r'^{-(m+2)}\left\{\frac{2m!\,m}{2^m.m!}-\frac{2m-2!(m-2)}{2^m.1!\,m-1!}(ihr')^2+\frac{2m-4!(m-4)}{2^m.2!\,m-2!}(ihr')^4\ldots\right\} \end{aligned}\right\}(6),$$

where the constants B_m, B_m' have to be chosen so as to satisfy the conditions

$$\left.\begin{aligned} u_n'+(u_n') &= 0 \\ v_n+(v_n') &= 0 \end{aligned}\right\},\ \text{when } r'=b.$$

These equations of condition will give

$$B_m.ik = (-1)^n.ic\frac{2n!(A_nk-nA_n'h)}{2^n.(n!)^2c^{n+3}}.\frac{2^{m-1}.m+n!}{2m-2!\{mk^2+(m+1)h^2\}}.\frac{b^{2m-1}}{c^{m-1}}$$
$$\left[2m+1-\frac{m(2m+1)k^4+4(m+1)h^4+(m+1)(2m-3)h^2k^2}{2(2m-3)\{mk^2+(m+1)h^2\}}b^2\right],$$

$$B_m'.ih = (-1)^n.ic\frac{2n!(A_nk-nA_n'h)}{2^n.(n!)^2c^{n+3}}.\frac{2^{m-1}.m+n!}{m.2m-2!\{mk^2+(m+1)h^2\}}.\frac{b^{2m-1}}{c^{m-1}}$$
$$\left[2m+1-\frac{(m+1)(2m+1)h^4+4mk^4+m(2m-3)h^2k^2}{2(2m-3)\{mk^2+(m+1)h^2\}}b^2\right],$$

to the second approximation. It is necessary to determine B_m and B_m' to this order of approximation, for, if we neglect the squares, it will be found on substitution that we get $(u_n')=0$, $(v_n')=0$.

Substituting in equations (6) and writing C_n for $(-1)^n ik.A_n$, C_n' for $(-1)^n.ih.nA_n'$, we get

$$\left.\begin{aligned} (u_n') &= \frac{2n!(C_n-C_n')}{2^n(n!)^2c^{n+2}}\sum_{m=0}^{m=\infty}(-1)^{m+1}P_m(\mu')\frac{m+n![(2m+1)\{(m-1)k^2-(m+1)h^2\}r'^2-(m+1)(2m-1)(k^2-h^2)b^2]}{2.m!\{mk^2+(m+1)h^2\}r'^{m+2}}\frac{b^{2m-1}}{c^{2m-1}}.e^{ipt} \\ (v_n') &= \frac{2n!(C_n-C_n')}{2^n(n!)^2c^{n+2}}\sum_{m=1}^{m=\infty}(-1)^m\frac{dP_m(\mu')}{d\theta'}.\frac{m+n![(2m+1)\{mk^2-(m-2)h^2\}r'^2-m(2m-1)(k^2-h^2)b^2]}{2.m!\{mk^2+(m+1)h^2\}r'^{m+2}}\frac{b^{2m-1}}{c^{2m-1}}e^{ipt} \end{aligned}\right\}(7).$$

The displacement, whose components are given in equations (7), will be that produced by the presence of B in the field of vibration, if high powers of kr and hr are neglected.

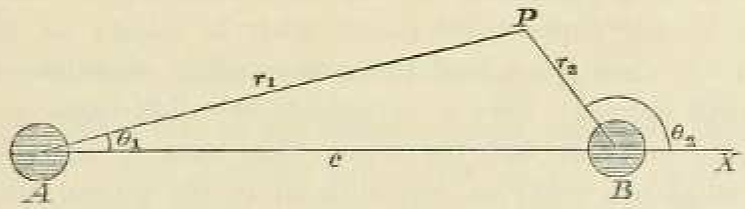
9. Let us now consider the displacement produced by the simultaneous pulsations of two small spheres in an elastic medium, the waves in which are long compared with the distance between the spheres; the centres of the spheres being supposed fixed in space, and the displacements such that no slipping takes place at the surfaces of the spheres.

Let the radii of the spheres at the time t be given by the equations

$$r_a = a(1 + u_a \sin pt),$$

$$r_b = b(1 + u_b \sin pt),$$

where u_a, u_b are small quantities.



The displacement at any point due to the pulsations of A alone, if we neglect the disturbance due to the presence of B in the field of vibration, will be compounded of

$$u_1 = \text{displacement along } AP = \frac{d}{dr_1}\{f_1(kr_1 - pt)\},$$

$$v_1 = \text{displacement perpendicular to } AP = 0,$$

where
$$f_1(kr_1 - pt) = A\,\frac{\cos(pt - kr_1 + \alpha)}{r_1};$$

A and α being determined by the boundary conditions.

Similarly the displacement due to the pulsations of B alone will be compounded of

$$u_2 = \text{displacement along } BP = \frac{d}{dr_2}\{f_2(kr_2 - pt)\},$$

$$v_2 = \text{displacement perpendicular to } BP = 0,$$

where
$$f_2(kr_2 - pt) = \frac{B\cos(pt - kr_2 + \beta)}{r_2};$$

B and β being determined by the boundary conditions.

Putting f_1 and f_2 into the exponential form, we have

$$\left.\begin{aligned} f_1 &= \frac{iA\cos\alpha - A\sin\alpha}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{-i(kr_1 - pt)} - \frac{iA\cos\alpha + A\sin\alpha}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{i(kr_1 - pt)} \\ f_2 &= \frac{iB\cos\beta - B\sin\beta}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{-i(kr_2 - pt)} - \frac{iB\cos\beta + B\sin\beta}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{i(kr_2 - pt)} \end{aligned}\right\} \ldots(1).$$

The equations to find the constants are obtained by expressing that u_1 must be $au_a \sin pt$ at the surface of A and u_2 $bu_b \sin pt$ at the surface of B for all values of t.

These conditions give

$$\left.\begin{aligned} A\,\{ka\cos(ka-\alpha)-\sin(ka-\alpha)\} &= a^3u_a \\ ka\sin(ka-\alpha)+\cos(ka-\alpha) &= 0 \\ B\,\{kb\cos(kb-\beta)-\sin(kb-\beta)\} &= b^3u_b \\ kb\sin(kb-\beta)+\cos(kb-\beta) &= 0 \end{aligned}\right\} \quad\text{..........................(2)},$$

whence the constants can be determined.

The functions u_1 and u_2 will give the whole displacement to a high degree of approximation at considerable distances from the spheres, the radii of which are supposed to be small. But it is evident that, in the immediate neighbourhood of the pulsating bodies, these values for the displacement cannot safely be taken; for, the surface conditions $u = au_a \sin pt$ and $v = 0$ when $r_1 = a$, and the corresponding conditions when $r_2 = b$ will not be satisfied. We shall therefore have to investigate the disturbance at the surfaces of the spheres A and B, in order to find the terms that have to be added to complete the solution. This investigation would be very difficult in the general case; but, since in the case under consideration we take the waves to be long, so that the lowest powers only of kr need to be retained in the neighbourhood of the vibrating bodies, we can by successive applications of equations (7), § 8, obtain a solution to any order of approximation that may be required.

Taking account only of the lower powers of kc, ka, kb; we shall add terms u_2', v_2' of the forms given in the equations just alluded to, so that at those points, whose original distance from the centre of B was b, the displacement may be compounded of $au_b \sin pt$ along the radius vector and zero perpendicular to it. The conditions at the surface of A will not be satisfied by these values, but, if we transform to the centre of A as pole, we can add terms which will satisfy the conditions at the surface of that body. This series of operations will in general have to be continued indefinitely; but, if an approximate solution only is required, and if it should appear that the surface conditions are satisfied for both bodies if a certain order of small quantities is rejected, we shall have obtained a complete solution to that order of approximation.

Neglecting all the powers of ka and kb except the lowest, we have, by (2),

$$A = a^3u_a, \quad B = b^3u_b, \quad \sin\alpha = 1, \quad \sin\beta = 1,$$

therefore equations (1) become

$$\left.\begin{aligned} f_1(kr_1 - pt) &= -\frac{a^3u_a}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{i(kr_1-pt)} - \frac{a^3u_a}{2k}\left(\frac{1}{r_1}\frac{d}{dr_1}\right)e^{-i(kr_1-pt)} \\ f_2(kr_2 - pt) &= -\frac{b^3u_b}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{i(kr_2-pt)} - \frac{b^3u_b}{2k}\left(\frac{1}{r_2}\frac{d}{dr_2}\right)e^{-i(kr_2-pt)} \end{aligned}\right\} \quad\text{................(3)},$$

whence we obtain to a first approximation

$$u_1 = \frac{a^3u_a}{2r_1^2}ie^{-ipt}, \quad u_2 = \frac{b^3u_b}{2r_2^2}ie^{-ipt} \quad\ldots\ldots\ldots(4),$$

with two other terms obtained by changing the sign of i.

Introducing the condition that, at all points which were when undisturbed on the surface of the sphere B, the displacement is to be bu_bie^{-ipt} along the radius vector and zero perpendicular to it; we shall find that, in order to find the whole disturbance, we shall have to introduce terms u_2', v_2' of the same form as those given in equation (7), § 8, so that

$$\left.\begin{aligned} u_2' &= ic\frac{a^3u_a}{2c^3}e^{-ipt}\Sigma(-1)^{n+1}P_n(\mu_2).\frac{n\left[(2n+1)\{(n-1)k^2-(n+1)h^2\}r_2^2-(n+1)(2n-1)(k^2-h^2)b^2\right]}{2\{nk^2+(n+1)h^2\}r_2^{n+2}}\frac{b^{2n-1}}{c^{n-1}} \\ v_2' &= ic\frac{a^3u_a}{2c^3}e^{-ipt}\Sigma(-1)^n\frac{dP_n(\mu_2)}{d\theta_2}.\frac{(2n+1)\{nk^2-(n-2)h^2\}r_2^2-n(2n-1)(k^2-h^2)b^2}{2\{nk^2+(n+1)h^2\}r_2^{n+2}}\frac{b^{2n-1}}{c^{n-1}} \end{aligned}\right\} \quad\ldots\ldots(5),$$

if $\mu_2 = \cos\theta_2$; θ_2 being the angle which r_2 makes with BX.

Again, using the condition that at the surface of A, u should be equal to au_aie^{-ipt} and that v should be zero; we shall have to add terms u_1', v_1' in order to counteract the disturbance caused by the displacement u_2 along BP. These terms will, as before, be

$$\left.\begin{aligned} u_1' &= -ic\frac{b^3u_b}{2c^3}e^{-ipt}\Sigma P_n(\mu_1).\frac{n\left[(2n+1)\{(n-1)k^2-(n+1)h^2\}r_1^2-(n+1)(2n-1)(k^2-h^2)a^2\right]}{2\{nk^2+(n+1)h^2\}r_1^{n+2}}\frac{a^{2n-1}}{c^{n-1}} \\ v_1' &= ic\frac{b^3u_b}{2c^3}e^{-ipt}\Sigma\frac{dP_n(\mu_1)}{d\theta_1}.\frac{(2n+1)\{nk^2-(n-2)h^2\}r_1^2-n(2n-1)(k^2-h^2)a^2}{2\{nk^2+(n+1)h^2\}r_1^{n+2}}\frac{a^{2n-1}}{c^{n-1}} \end{aligned}\right\} \quad\ldots\ldots(6).$$

The disturbance produced by u_2', v_2' at the surface of A and by u_1', v_1' at the surface of B will be found to be of the third order of the small quantities $\frac{a}{c}$ and $\frac{b}{c}$. If therefore we neglect the terms of the third order, we shall have a complete solution for the displacement at any point. This displacement is given by those parts of the terms u_1, u_1', v_1'; u_2, u_2', v_2' which do not involve small quantities of the third order.

At the surface of A the displacement will consist of

$$\left.\begin{aligned} u &= \text{displacement along } AP = \frac{a^3u_a}{2r_1^2}ie^{-ipt} - \frac{b^3u_b}{2c^2}P_1(\mu_1)ie^{-ipt} + \frac{ab^3u_b}{2c^2}.\frac{3h^2r_1^2+(k^2-h^2)a^2}{(k^2+2h^2)r_1^3}P_1(\mu_1)ie^{-ipt} \\ v &= \text{displacement perpendicular to } AP \\ &= -\frac{b^3u_b}{2c^2}\frac{dP_1(\mu_1)}{d\theta_1}ie^{-ipt} + \frac{ab^3u_b}{2c^2}.\frac{3(k^2+h^2)r_1^2-(k^2-h^2)a^2}{2(k^2+2h^2)r_1^3}\frac{dP_1(\mu_1)}{d\theta_1}ie^{-ipt} \end{aligned}\right\} \quad\ldots\ldots(7),$$

with similar terms obtained by changing the sign of i.

Similarly the displacement at the surface of B can be obtained.

10. *Investigation of the mutual action of two small spheres pulsating in an elastic medium, the waves being supposed to be long compared with the distance between the centres of the spheres.*

Let us suppose the centres to be fixed in space, and assume that no slipping takes place at the surfaces. Under these circumstances the displacement can be found as in § 9, and if small quantities of the third and higher orders be neglected we shall have for the components of the displacement at the surface of A,

$$u = \frac{a^3 u_a}{2r_1^2} ie^{-ipt} - \frac{b^3 u_b}{2c^2} P_1(\mu_1) ie^{-ipt} + \frac{ab^3 u_b}{2c^2} \cdot \frac{3h^2 r_1^2 + (k^2 - h^2) a^2}{(k^2 + 2h^2) r_1^3} P_1(\mu_1) ie^{-ipt},$$

$$v = -\frac{b^3 u_b}{2c^2} \frac{dP_1(\mu_1)}{d\theta_1} ie^{-ipt} + \frac{ab^3 u_b}{2c^2} \frac{3(k^2 + h^2) r_1^2 + (k^2 - h^2) a^2}{2(k^2 + 2h^2) r_1^3} \frac{dP_1(\mu_1)}{d\theta_1} ie^{-ipt},$$

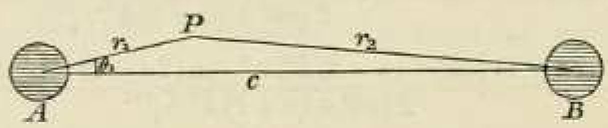
with two similar terms got by changing sign of i.

When $r_1 = a$ we shall have

$$u = \frac{au_a}{2} ie^{-ipt}, \quad v = 0, \quad \frac{du}{d\theta_1} = 0, \quad \frac{dv}{d\theta_1} = 0, \text{ and hence the dilatation } \epsilon = \frac{du}{dr_1}.$$

Also the force on the sphere A resolved along AB in the direction AB

$$= \int\left\{\left(\lambda\epsilon + 2\mu \frac{du}{dr_1}\right)\cos\theta_1 - \mu\left(\frac{1}{r_1}\frac{du}{d\theta_1} + \frac{dv}{dr_1} - \frac{v}{r_1}\right)\sin\theta_1\right\} d\sigma,$$



where $d\sigma$ is an element of the surface and the integration is performed over the whole of the sphere.

Since $\int P_n(\mu_1)\cos\theta_1 d\sigma = 0$ and $\int \frac{dP_n(\mu_1)}{d\theta_1}\sin\theta_1 d\sigma = 0$, unless $n = 1$, we need only attend to the terms in u and v involving $P_1(\mu_1)$ and $\frac{dP_1(\mu_1)}{d\theta_1}$.

Hence the force in the direction AB

$$= -\frac{3b^3 u_b}{2ac^2} \cdot ie^{-ipt} \int_0^\pi \frac{(\lambda + 2\mu) k^2 \cos^2\theta_1 + \mu h^2 \sin^2\theta_1}{k^2 + 2h^2} 2\pi r_a^2 \sin\theta d\theta.$$

Introducing the condition $(\lambda + 2\mu) k^2 = \mu h^2$ and taking account of the double sign of i, we get

$$\text{Force in direction } AB \text{ at the time } t = -\frac{12\pi\mu(\lambda + 2\mu)}{2\lambda + 5\mu} \frac{b^3}{ac^2} \cdot r_a^2 u_b \sin pt \ldots\ldots\ldots(1).$$

The resultant force on A can be found if we know the forms of u_a, u_b. If $u_a = \rho_1$, $u_b = \rho_2$, where ρ_1 and ρ_2 are constants, we shall have during a complete period $\frac{2\pi}{p}$

Force on A in direction $AB = -\dfrac{12\pi\mu(\lambda+2\mu)}{2\lambda+5\mu}\dfrac{b^3}{ac^2}\displaystyle\int_0^{\frac{2\pi}{p}} a^2\rho_2(1+\rho_1\sin pt)^2\sin pt\,dt$

$$= -\frac{24\pi^2\mu(\lambda+2\mu)}{p(2\lambda+5\mu)}\rho_1\rho_2\frac{b^3a}{c^2} \quad\text{..................(2).}$$

This is a repulsion when ρ_1 and ρ_2 have the same sign and an attraction if they have opposite signs. In either case the force varies according to the inverse square of the distance between A and B.

11. *Second approximation.*

We shall now include the terms of the third order in the expression for the displacement. As in § 10, it is evident that the only terms which will enter into the expression for the force on A will be the coefficients of $P_1(\mu_1)$, and of $\dfrac{dP_1(\mu_1)}{d\theta_1}$. The coefficients of these terms in u_1, u_2, u_1', v_1' have already been considered, and the action arising from them has been given in equation (2), § 10. We have next to resolve the displacement given by u_2', v_2' in equations (5), § 9 along and perpendicular to AP. If the resolved components are (u_2') and (v_2') we get, if the terms of the fourth order are neglected,

$$\left.\begin{aligned}(u_2') &= \frac{iba^3u_a}{2c}e^{-ipt}\left\{\left(\cos\theta_1-\frac{r_1}{c}\right)P_1(\mu_2)\cdot\frac{3h^2r_2^2+(k^2-h^2)b^2}{(k^2+2h^2)r_2^4}+\sin\theta_1\frac{dP_1(\mu_2)}{d\theta_2}\cdot\frac{3(h^2+k^2)r_2^2-(k^2-h^2)b^2}{2(k^2+2h^2)r_2^4}\right\}\\ (v_2') &= -\frac{iba^3u_a}{2c}e^{-ipt}\left\{\sin\theta_1P_1(\mu_2)\frac{3h^2r_2^2+(k^2-h^2)b^2}{(k^2+2h^2)r_2^4}+\left(\cos\theta_1-\frac{r_1}{c}\right)\frac{dP_1(\mu_2)}{d\theta_2}\cdot\frac{3(h^2+k^2)r_2^2-(k^2-h^2)b^2}{2(k^2+2h^2)r_2^4}\right\}\end{aligned}\right\}\ (1).$$

These terms can be expanded in series involving Legendre's coefficients of μ_1 and their differentials with respect to θ_1. For by equation (3), § 8, we get

$$\frac{P_n(\mu_2)}{r_2^{n+1}} = (-1)^n\cdot\frac{1}{c^{n+1}}\left\{1+(n+1)P_1(\mu_1)\frac{r_1}{c}+\frac{(n+1)(n+2)}{2!}P_2(\mu_1)\frac{r_1^2}{c^2}+\ldots\right\} \text{......(2),}$$

therefore

$$\frac{\sin\theta_1\cdot P_n(\mu_2)}{r_2^{n+1}} = (-1)^{n+1}\cdot\frac{1}{c^{n+1}}\left[\frac{dP_1(\mu_1)}{d\theta_1}+\frac{n+1}{3}\frac{dP_2(\mu_1)}{d\theta_1}\frac{r_1}{c}\right.$$
$$\left.+\frac{(n+1)(n+2)}{5\cdot 2!}\left\{\frac{dP_3(\mu_1)}{d\theta_1}-\frac{dP_1(\mu_1)}{d\theta_1}\right\}\frac{r_1^2}{c^2}+\text{etc.}\right] \text{.........(3),}$$

and
$$\frac{P_n(\mu_2)}{r_2^{n+1}}\left(\cos\theta_1-\frac{r_1}{c}\right) = (-1)^n\cdot\frac{1}{c^{n+1}}\left[P_1(\mu_1)+\frac{2(n+1)P_2(\mu_1)-(n-2)P_0(\mu_1)}{3\cdot 1!}\frac{r_1}{c}\right.$$
$$\left.+\frac{n+1}{5\cdot 2!}\{3(n+2)P_3(\mu_1)+2(n-3)P_1(\mu_1)\}\frac{r_1^2}{c^2}+\ldots\right] \text{............ (4).}$$

Also differentiating equation (2) with respect to θ_1 keeping r_2 constant, we get

$$\frac{1}{r_2^{n+1}}\frac{dP_n(\mu_2)}{d\theta_2} = (-1)^n\cdot\frac{n(n+1)r_1}{c^{n+2}}\left\{\frac{1}{2!}\frac{dP_1(\mu_1)}{d\theta_1}+\frac{n+2}{3!}\frac{dP_2(\mu_1)}{d\theta_1}\frac{r_1}{c}\right.$$
$$\left.+\frac{(n+2)(n+3)}{4!}\frac{dP_3(\mu_1)}{d\theta_1}\frac{r_1^2}{c^2}+\ldots\right\} \text{......................(5),}$$

therefore $$\frac{\sin\theta_1}{r_2^{n+1}}\frac{dP_n(\mu_2)}{d\theta_2} = (-1)^n.\frac{n(n+1)r_1}{c^{n+2}}\left[\frac{1}{3.1!}\{P_2(\mu_1)-P_0(\mu_1)\}\right.$$
$$\left.+\frac{n+2}{5.2!}\{P_3(\mu_1)-P_1(\mu_1)\}\frac{r_1}{c}+\frac{(n+2)(n+3)}{7.3!}\{P_4(\mu_1)-P_2(\mu_1)\}\frac{r_1^2}{c^2}+\ldots\right]\ldots\ldots\ldots(6),$$

and $$\frac{1}{r_2^{n+1}}\left(\cos\theta_1-\frac{r_1}{c}\right)\frac{dP_n(\mu_2)}{d\theta_2} = (-1)^n.\frac{n(n+1)r_1}{c^{n+2}}\left[\frac{1}{3.2!}\frac{dP_2(\mu_1)}{d\theta_1}\right.$$
$$\left.+\frac{1}{5.3!}\left\{2(n+2)\frac{dP_3(\mu_1)}{d\theta_1}+3(n-3)\frac{dP_1(\mu_1)}{d\theta_1}\right\}\frac{r_1}{c}+\frac{n+2}{7.4!}\left\{3(n+3)\frac{dP_4(\mu_1)}{d\theta_1}+4(n-4)\frac{dP_2(\mu_1)}{d\theta_1}\right\}\frac{r^2}{c^2}\ldots\right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

If all the terms of a higher order than the third be neglected, we get by the above equations

$$\left.\begin{aligned}(u_2') &= -\frac{iba^3u_a}{2c^3}e^{-ipt}\frac{3h^2}{k^2+2h^2}P_1(\mu_1)\\ (v_2') &= -\frac{iba^3u_a}{2c^3}e^{-ipt}\frac{3h^2}{k^2+2h^2}\frac{dP_1(\mu_1)}{d\theta_1}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Using equation (7), § 8, we find the terms which have to be added in order to make the whole displacement equal to au_aie^{-ipt} at the surface of A. They are

$$\left.\begin{aligned}u_1'' &= \frac{iba^6u_a}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+2h^2}\left\{\frac{3h^2}{(k^2+2h^2)a^2r_1}+\frac{k^2-h^2}{(k^2+2h^2)r_1^3}\right\}P_1(\mu_1)\\ v_1'' &= \frac{iba^6u_a}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+3h^2}\left\{\frac{3(h^2+k^2)}{2(k^2+2h^2)a^2r_1}-\frac{k^2-h^2}{2(k^2+2h^2)r_1^3}\right\}\frac{dP_1(\mu_1)}{d\theta_1}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Similarly, if we consider the whole displacement at the surface of B we can see that we shall have to add terms u_2'', v_2'' in order to satisfy the surface conditions to the third order of small quantities, where

$$\left.\begin{aligned}u_2'' &= -\frac{iab^6u_b}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+2h^2}\left\{\frac{3h^2}{(k^2+2h^2)b^2r_2}+\frac{k^2-h^2}{(k^2+2h^2)r_2^3}\right\}P_1(\mu_2)\\ v_2'' &= \frac{iab^6u_b}{2c^3}e^{-ipt}.\frac{3h^2}{k^2+2h^2}\left\{\frac{3(h^2+k^2)}{2(k^2+2h^2)b^2r_2}-\frac{k^2-h^2}{2(k^2+2h^2)r_2^3}\right\}\frac{dP_1(\mu_2)}{d\theta_2}\end{aligned}\right\}\ldots\ldots\ldots\ldots(10).$$

The effect of u_1'', v_1'' at the surface of B or of u_2'', v_2'' at the surface of A will be of the fourth order of small quantities, and can therefore be neglected to our approximation. Hence we have the complete expression for the coefficients of $P_1(\mu_1)$ and $\frac{dP_1(\mu_1)}{d\theta_1}$ to the third order of approximation.

The second term in the expression for the force on A may now be found as in § 10. It will be

$$-\frac{72\pi^2\mu(\lambda+2\mu)^2}{p(2\lambda+5\mu)^2}\rho_1\frac{{}_2ba^4}{c^3},$$

in the direction AB, if u_a and u_b are, as before, supposed to be constants. This gives a repulsive force, varying with the inverse cube of the distance between A and B.

12. If any explanations of the nature of gravitation, or of electrical forces, are to be based on suppositions of the transmission of motion, or stress through a medium intervening between the electrified bodies; it appears that such explanations must be based on the hypothesis that the medium behaves like an elastic solid, unless space is to be supposed to be filled with a second medium, whose properties differ from that in which light is supposed to be propagated. Thus it appears that hypotheses based on the theory of incompressible fluids must be erroneous; and it should be observed that even if two media are admitted, the second, if a fluid, can hardly be supposed to be absolutely incompressible. This appears to be a serious objection to a theory which has been adduced* to explain gravitation, based upon Bjerknes' investigation, in which it is supposed that all atoms are pulsating in phases not differing from one another by more than a quarter period, and that the intervening medium is an incompressible fluid. If this were the case, the law of universal attraction according to the law of the inverse square would follow; but unless the medium is supposed to be absolutely incompressible, in which case all pulsations would be instantaneously diffused throughout space, there would on this theory be repulsion between bodies at distances greater than a quarter wave length, and bodies would at certain distances repel one another, which is contrary to observation.

The electrical phenomena of attraction and repulsion are capable of explanation by the results proved in this paper. If we suppose two groups of particles to be pulsating, so that the phases of the particles in the first group lie between $\frac{p}{4}$ and $-\frac{p}{4}$, and that those of the other group lie between $\frac{3p}{4}$ and $-\frac{3p}{4}$; $2p$ being the complete period; we shall find, by an obvious extension of equation (2), § 10, that the particles of the same group will repel each other, the law of force being that of the inverse square of the distance, and will attract the particles belonging to the other group according to the same law of force. This is in accordance with observed electrical actions, and is based on the hypothesis that the medium which transmits electrical vibrations is the same as that by which the undulations of light are transmitted. The observed electrical effects of attraction and repulsion, conduction and induction, can be explained on this hypothesis, as in the ordinary two fluid theory. It should however be observed that the laws of attraction and repulsion are supposed to hold good only for distances which do not exceed a quarter wave length of the displacement corresponding to the period of the particles' pulsation. This is as yet unverified by experiment, but, if it be untrue, all hypotheses of electrical action arising from supposed periodic disturbances of the intervening medium owing to the electrical condition of the body will in like manner be vitiated. It should also be noticed that no hypothesis is made concerning the shape or density or other properties of the two groups of particles, which may be supposed to differ in any or all of these respects:—the only suppositions made concerning them are that they are pulsating, and that their phases satisfy the conditions above indicated. Several other electrical phenomena may be explained by a hypothesis similar to that given above. If we suppose the

* *Proceedings of the Cambridge Philosophical Society*, Vol. III. Part VII.

atoms composing matter to be also pulsating, then, by proper suppositions with regard to their phases, we can explain the cohesion of two or more atoms to form a molecule, the apparent comparative affinity for one or the other kind of electricity that bodies appear to possess, and several of the phenomena of electrolysis.